Chapter 3 Theory of Angular Momentum

* Review : Continuous Symmetry in CM and QM, so far...

"Continuous Symmetry" [Operations]

$\leftarrow$ a family of "infinitesimal" (CM : canonical transformations / QM : unitary transformations)

$\Longrightarrow$ "Lie" groups.

▽ Brief review of the canonical transformations in CM.
(see David Tong's lecture note for more details)

def.

$$\left[\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned}\right] \xrightarrow[\left(\begin{aligned} Q_i &\equiv Q_i(\vec{q},\vec{p}) \\ P_i &\equiv P_i(\vec{q},\vec{p}) \end{aligned}\right)]{\text{Canonical transformation}} \left[\begin{aligned} \dot{Q}_i &= \frac{\partial H}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial H}{\partial Q_i} \end{aligned}\right]$$

$\rightarrow$ Invariance of the Hamilton's EOM.

- In a more abstract form,

$$\text{EOM}: \quad \dot{\vec{x}} = J\frac{\partial H}{\partial \vec{x}} . \qquad \Bigg\| \quad \vec{x} = (q_1 \cdots q_n, p_1 \cdots p_n)^T, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- Transformations $q_i \rightarrow Q_i(\vec{q},\vec{p})$, $p_i \rightarrow P_i(\vec{q},\vec{p})$

can be now rewritten as

$$x_i \longrightarrow y_i(\vec{x}) \qquad \Big\| \quad \vec{y} = (\vec{Q}, \vec{P})$$

- new EOM :

$$\dot{y}_i = \frac{\partial y_i}{\partial x_j}\dot{x}_j = \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial H}{\partial x_k} \qquad \Big| = \frac{\partial H}{\partial y_l}\frac{\partial y_l}{\partial x_k}$$

$$\Longrightarrow \quad \dot{y}_i = \left(\frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial y_l}{\partial x_k}\right) \frac{\partial H}{\partial y_l}$$

$$\Rightarrow \dot{\vec{y}} = \mathcal{J} J \mathcal{J}^T \frac{\partial H}{\partial \vec{y}}$$ in a matrix form

$\| \; \mathcal{J}_{ij} = \frac{\partial y_i}{\partial x_j}$

Since the canonical transformations do not change the EOM : $\dot{\vec{y}} = J \frac{\partial H}{\partial \vec{y}}$.

$\Rightarrow$ Requirement of the canonical transformations

: $\boxed{\mathcal{J} J \mathcal{J}^T = J}$

$\| \; [A, B]_{P.B} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$.

$\rightarrow$ This is, in fact, equivalent to

$$[Q_i, Q_j]_{P.B.} = [P_i, P_j]_{PB} = 0$$

and $[Q_i, P_j]_{PB} = \delta_{ij}$: The invariance of the Poisson brackets.

One can directly see this by using the def. of $\mathcal{J}_{ij}$,

$$\mathcal{J}_{ij} = \frac{\partial y_i}{\partial x_j} \doteq \left(\begin{array}{c|c} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \hline \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{array} \right),$$

and putting it into $\mathcal{J} J \mathcal{J}^T = J$.

- Infinitesimal canonical transformations

$$\left(\begin{array}{l} q_i \rightarrow Q_i = q_i + \alpha F_i(\vec{q}, \vec{p}) \\ p_i \rightarrow P_i = p_i + \alpha E_i(\vec{q}, \vec{p}) \end{array} \right.$$ for small α.

$$\Downarrow \; \mathcal{J} J \mathcal{J}^T = J$$

$$\left(\begin{array}{l} q_i \rightarrow Q_i = q_i + \alpha \frac{\partial G}{\partial p_i} \\ p_i \rightarrow P_i = p_i - \alpha \frac{\partial G}{\partial q_i} \end{array} \right.$$

$\| \; G = G(\vec{q}, \vec{p})$ "Generating Function"

In other words, w.r.t. the infinitesimal α,

$$\left(\begin{array}{l} Q_i = q_i + \frac{dq_i}{d\alpha}\cdot\alpha \\ P_i = p_i + \frac{dp_i}{d\alpha}\cdot\alpha \end{array} \right) \Rightarrow \left(\begin{array}{l} \frac{dq_i}{d\alpha} = \frac{\partial G}{\partial p_i} \\ \frac{dp_i}{d\alpha} = -\frac{\partial G}{\partial q_i} \end{array} \right) \; !$$

- Infinitesimal change in an observable $A \equiv A(\vec{q}, \vec{p})$

$$\delta A = \frac{\partial A}{\partial q_i}\delta q_i + \frac{\partial A}{\partial p_i}\delta p_i$$

$$= \frac{\partial A}{\partial q_i}\cdot\alpha\frac{dq_i}{d\alpha} + \frac{\partial A}{\partial p_i}\alpha\frac{dp_i}{d\alpha}$$

$$= \alpha\frac{\partial A}{\partial q_i}\frac{\partial G}{\partial p_i} - \alpha\frac{\partial A}{\partial p_i}\frac{\partial G}{\partial q_i}$$

$$\Rightarrow \underline{\delta A = \alpha [A, G]_{P.B}} \; // \qquad \Big\| \; \text{def. } [A,B]_{P.B} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}.$$

Ex. spatial translation — The Generator !

$$\begin{array}{l} q_i \to Q_i = q_i + \alpha \\ p_i \to P_i = p_i \end{array} \Rightarrow G = \underline{p_i}$$

and $\delta A = \alpha [A, p_i]_{P.B}$ $\quad \| \; \delta A = A(q_i+\alpha) - A(q_i)$

- Time-Evolution of an observable.

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i}\cdot\dot{q}_i + \frac{\partial A}{\partial p_i}\cdot\dot{p}_i + \frac{\partial A}{\partial t} \qquad \dot{\vec{x}} = J\frac{\partial H}{\partial \vec{x}} \quad \text{(EOM)}$$

$$= \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\cdot\frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial t}$$

$$\Rightarrow \underline{\frac{dA}{dt} = [A, H]_{P.B.}} \; // \quad \longleftrightarrow \text{ Heisenberg EOM.}$$

$$+ \frac{\partial A}{\partial t} \qquad \left([\,,\,]_{PB} \to \frac{1}{i\hbar}[\,,\,] \right)$$

- Noether's theorem.

a Continuous symmetry $\longrightarrow$ an integral of motion (conserved quantity)

: $\delta H = \alpha\, [H, G]_{P.B}$.

$\oplus$ $\dfrac{dG}{dt} = [G, H]_{P.B} + \dfrac{\partial G}{\partial t}$

$\Rightarrow$ If H is invariant under a certain cont. sym. operation, its generator is preserved.

ex. spatial translational invariance : $G = p$

$\delta H = H(q+\alpha) - H(q) = 0$.

$\rightarrow$ $[H, p] = 0$ $\Rightarrow$ $\dfrac{dp}{dt} = 0$.

linear momentum conservation.

ex. time translation invariance : $G = H$

$\delta H = H(t+\alpha) - H(t) = 0$.

: $[H, H] = 0$, $\dfrac{\partial H}{dt} = 0$. ($t$-indep. H)

$\Rightarrow$ $\dfrac{dH}{dt} = 0$: Energy conservation.

- Continuous symmetry in QM, so far ...

We have seen

1) spatial translation
$$\mathcal{T}(a) = \exp\left[-\tfrac{i}{\hbar}\hat{p}a\right]$$

2) time evolution
$$\mathcal{U}(t) = \exp\left[-\tfrac{i}{\hbar}Ht\right]$$

← The key property that we have used to derive these:

$$\mathcal{U}(\alpha_1 + \alpha_2) = \mathcal{U}(\alpha_1)\cdot\mathcal{U}(\alpha_2) \quad \cdots (*)$$

|| "Abelian" group property.

⊕

The Stone theorem

Given a set of unitary operators depending on a continuous parameter α and satisfying the Abelian group law, there exists a Hermitian operator G, called the infinitesimal generator of the transformation group $\mathcal{U}(\alpha)$, such that $\mathcal{U}(\alpha) = e^{i\alpha G}$.

$u = e^{-\frac{i}{\hbar}\alpha G}$

- infinitesimal transformation can be written, when $(*)$ holds,

$$\mathcal{U}(\alpha) \simeq 1 - \frac{i}{\hbar}\alpha G + O\left[(\alpha/\hbar)^2\right]$$

- The corresponding change of an observable:

$$F(a+\delta a) = \mathcal{U}^{+} F(a)\,\mathcal{U}$$

$$= F(a) + \frac{i}{\hbar}\delta a\,[G, F] + \cdots$$

$$\Rightarrow \delta F = \frac{i}{\hbar}\delta a\,[G, F] \quad \text{or} \quad \frac{\delta F}{\delta a} = \frac{i}{\hbar}[G, F]$$

! "classical - quantum correspondence" !

ex. spatial translation.

If $G = \tilde{P}$,

① $\delta A = \alpha[A, \tilde{p}]_{PB} \Longleftrightarrow \delta F = \frac{i}{\hbar}\delta a\,[\tilde{p}, F]$

[classical] [Quantum]

② $\lim_{\delta a \to 0} \frac{\delta F}{\delta a} = \frac{\partial F}{\delta x} = \frac{i}{\hbar}[\tilde{p}, F]$

$\Rightarrow \quad [\tilde{p}, F(\tilde{x})] = -i\hbar \frac{\partial}{\partial \tilde{x}} F$

If $F(\tilde{x}) = \tilde{x}$, $\quad [\tilde{x}, \tilde{p}] = i\hbar$.

ex. time-evolution.

$\delta A = \delta t\, [A, H]_{P.B.} \longrightarrow \delta A = \frac{i}{\hbar} \delta t\, [H, A]$

$\Rightarrow \quad \frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$: Heisenberg EOM.

Thus.

"G" is the same!

Classical $\longrightarrow$ Quantum. $\quad$ "$e^{-\frac{i}{\hbar}\alpha G}$"

Canonical transformation $\quad$ Unitary transformation.

$\left(\begin{array}{l} Q_i = q_i + \alpha \frac{\partial G}{\partial p_i} \\ P_i = p_i - \alpha \frac{\partial G}{\partial q_i} \end{array} \right)$ $\qquad$ $\left(1 - \frac{i}{\hbar}\alpha G \right)$

(1) Rotations in C.M. and Q.M.

The trouble! : $\quad U(\alpha_1)\, U(\alpha_2) \neq U(\alpha_2)\, U(\alpha_1)$

"non-Abelian"

$$e^{-\frac{i}{\hbar}\alpha_1 G_1}\, e^{-\frac{i}{\hbar}\alpha_2 G_2} = \exp\left[-\frac{i}{\hbar}(\alpha_1 G_1 + \alpha_2 G_2) \right]$$

only when $[G_1, G_2] = 0$.

$\Rightarrow$ This is broken in general for Rotations.

(in Both of C.M. and Q.M.).

*NOTE : We're talking about "3D" here.